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Quantum mechanics with q-deformed commutators and periodic variables

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Abstract. A q-deformed commutator for arbitrary q is derived from a variable with a periodic boundary condition such as an azimuthal angle φ ($0 \leqslant \varphi < 2\pi$). A Hamiltonian can be written down in an Hermitian form for $q = e^{\alpha}$ or $q = e^{i\alpha}$ with $\alpha \in \boldsymbol{R}$, and its eigenfunctions and eigenvalues are obtained. Algebraic structures, $W_{1+\infty}$ and $U_q(sl_2)$, of this model and introductions of gauge interactions are discussed. Extensions to many variables and some elementary examples are presented.

1. Introduction

Models with quantum groups or q-deformed commutator are widely discussed in a variety of contexts, for instance, q-deformed oscillator systems [1–3] will bring us to q-deformed quantum field theory and quantum mechanics based on non-commutative differential geometry [4–9] will shed light on the quantization of space-time. They are conceptually very interesting but are not yet realistic enough to observe the q-deformed effects in real physical processes. Recently we have presented quantum mechanics on a circle with a q-deformed commutation relation and pointed out some characteristic features which arise from the finiteness and the infinite degeneracy in the energy spectrum [10]. In the model the deformation parameter q was taken as a kth primitive root of unity, i.e. $q = \exp(i2\pi/k)$ with $k = 2, 3, \ldots$. We can immediately see that the q-deformed commutator given by

$$[e^{i\varphi}, p_{\varphi}^{(-)}]_q = i\hbar \tag{1.1}$$

is satisfied for arbitrary complex number q, where $[A, B]_q \equiv qAB - BA$ and

$$p_{\varphi}^{(-)} = -i\hbar e^{-i\varphi} \frac{1 - q^{-i\partial_{\varphi}}}{1 - q} \tag{1.2}$$

with $\partial_{\varphi} \partial/\partial\varphi$. Generally speaking, $\hbar$ in (1.1) and (1.2) may be replaced by arbitrary functions of q. We may also guess that the above model will be able to apply to motions for the azimuthal angle variable in two- or three-dimensional problems and some other physical processes with periodic boundary conditions. Furthermore, we may have a question whether this model can be extended to models with many periodic variables. We should also study more precisely the meaning of the character-

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istic features pointed out in the previous work, that is, the finiteness and the infinite degeneracy of the energy levels. As was noted in [10], these are closely related to the infinite dimensional symmetry, $W_{1+\infty}$ [11], of the model. In this paper, we shall study these problems and give some answers. We also discuss q-deformed phenomena in some elementary processes.

In section 2 the general framework and solutions are presented. An introduction of $U(1)$ gauge interaction is performed in section 3. In section 4 some algebraic structures of this model are discussed and the relation of the infinite degeneracy in the energy spectrum with the symmetry of the model is clarified. In section 5 an extension to many variables are studied, and some elementary examples for azimuthal motions in two- and three-dimensional potential problems are investigated in section 6. Some remarks are presented in section 7.

2. General formalism and solutions

Following the work of [10], we start with the following q-deformed commutation relation:

$$[\mathrm{e}^{i\varphi}, p_\varphi^{(-)}]_q = ih(q) \tag{2.1}$$

where $p_\varphi^{(-)}$ is defined by

$$p_\varphi^{(-)} = -\mathrm{i}h(q)\mathrm{e}^{-\mathrm{i}\varphi}\frac{1-q^{-i\partial_\varphi}}{1-q}.^{1} \tag{2.2}$$

In (2.1) and (2.2), we may regard φ as an angle variable ($0 \leqslant \varphi < 2\pi$). The function $h(q)$ can be chosen arbitrarily as far as it reduces to Planck's constant $\hbar$ in the limit $q \to 1$:

$$\lim_{q\to 1} h(q) = \hbar. \tag{2.3}$$

The condition (2.3) guarantees that the system we are considering should turn back to ordinary quantum mechanics. By introducing complex variable $z = \mathrm{e}^{\mathrm{i}\varphi}$ we easily see that

$$p_\varphi^{(-)} f(\mathrm{e}^{\mathrm{i}\varphi}) = -\mathrm{i}h(q)\frac{f(z)-f(qz)}{z(1-q)}. \tag{2.4}$$

Thus, the momentum operator $p_\varphi^{(-)}$ generates a finite displacement of the coordinate z on the complex plane. The cordinate displacement corresponds to the shift of the angle $\varphi \to \varphi + \alpha$ when $q = \mathrm{e}^{\mathrm{i}\alpha}$, and to the dilatation $z \to qz$ when $q = \mathrm{e}^{\alpha}$ for $\alpha \in \boldsymbol{R}$. The q-deformed commutation relation (2.1) reduces to the usual commutation relation as

[1] This expression of a q-deformed momentum operator reminds us of that presented in [12]. Indeed, these two coincide after some replacements, although the variable ξ in [12] corresponding to the variable φ in this article was not necessarily periodic. The effects of q-deformation arise from the introduction of internal coordinate ξ.

$[e^{i\varphi}, p_\varphi^{(-)}]_q = ih(q) \overset{q\to 1}{\to} [z, p_z] = i\hbar$ with $p_z = -i\hbar\partial_z$.

Let us give a Hamiltonian as

$$H_q = \kappa p_\varphi^{(+)} p_\varphi^{(-)} \tag{2.5}$$

where

$$p_\varphi^{(+)} = ih^*(q)\frac{1-q^{i\partial_\varphi}}{1-q^{-1}} e^{i\varphi}. \tag{2.6}$$

When we rewrite the Hamiltonian explicitly as

$$H_q = \kappa |h(q)|^2 \frac{2-q^{i\partial_\varphi}-q^{-i\partial_\varphi}}{(1-q)(1-q^{-1})} \tag{2.7}$$

it is easily seen that the Hamiltonian is Hermite if and only if $q = e^\alpha$ or $q = e^{i\alpha}$ with $\alpha \in \boldsymbol{R}$. In the limit $q \to 1$, the Hamiltonian coincides with the well known Hamiltonian for a rotator with the moment of inertia $I = 1/2\kappa$ as [10]

$$H_I \equiv \lim_{q\to 1} H_q = -\frac{\hbar^2}{2I}\frac{\partial^2}{\partial\varphi^2}. \tag{2.8}$$

Eigenfunctions and eigenvalues of H_q are easily derived as follows:

$$H_q \Phi_m(\varphi) = E_m \Phi_m(\varphi) \tag{2.9}$$

where

$$E_m = \kappa|h(q)|^2 \frac{(1-q^m)(1-q^{-m})}{(1-q)(1-q^{-1})} \tag{2.10}$$

for

$$\Phi_m(\varphi) = \frac{1}{\sqrt{2\pi}} e^{im\varphi} \tag{2.11}$$

with $m \in Z$. In (2.9)–(2.11), the periodic boundary condition $\Phi_m(\varphi + 2\pi) = \Phi_m(\varphi)$ is postulated. Explicitly we can rewrite E_m as

$$\begin{aligned} E_m &= \kappa|h(q)|^2 \frac{\sinh^2(m\alpha/2)}{\sinh^2(\alpha/2)} \qquad \text{for } q = e^\alpha \\ E_m &= = \kappa|h(q)|^2 \frac{\sin^2(m\alpha/2)}{\sin^2(\alpha/2)} \qquad \text{for } q = e^{i\alpha} \end{aligned} \tag{2.12}$$

where α is taken to be real. Note that the eigenvalues given in (2.10) coincide with those of the usual rotator as

$$E_m \overset{q\to 1}{\to} \frac{\hbar^2}{2I} m^2.$$

From (2.12), we obtain the relation

$$E_m \geqslant 0 \tag{2.13}$$

and the symmetry under the replacement $m \to -m$, that is

$$E_m = E_{-m}. \tag{2.14}$$

In addition to (2.14), when the deformation parameter q is a root of unity $q=\exp(2\pi i/k)$ $(k=2,3,\ldots)$, the second equation in (2.12) indicates further symmetry which is called Z_k-symmetry. Indeed all the states described by Φ_{m+nk} with $n\in Z$ have the same energy E_m. We will see in section 4 that the symmetry is related to the algebra of infinite dimension, $W_{1+\infty}$. Owing to the symmetries, (2.14) and Z_k, we obtain remarkable facts; the number of energy level is finite, $E_0, E_1, \ldots, E_{k-1}$, and each energy level has an infinite degeneracy. Since we have already discussed some interesting phenomena in [10], we do not repeat them here.

3. Introduction of gauge interactions

In this section, we investigate gauge interactions and give some possible ways of introductions of gauge interaction. We first consider a local $U(1)$ gauge transformation for the eigenfunctions given in (2.11), that is, $\Phi_m(\varphi)\to\tilde{\Phi}_m(\varphi)=\Gamma_\alpha\Phi_m(\varphi)$, where

$$\Gamma_\alpha=\exp\left(\frac{e}{c\hbar}\alpha_q(\varphi)\right)\in U(1)$$

and e is the electric charge of the rotator. As in the $q=1$ case, $H_q\Phi_m(\varphi)$ does not transform covariantly under the local $U(1)$ gauge transformation, that is, $\Gamma_\alpha H_q\Phi_m(\varphi)\neq H_q\tilde{\Phi}_m(\varphi)$. So we have to introduce gauge potential A_φ and claim that

$$\Gamma_\alpha H_q^e\Phi_m(\varphi)=\tilde{H}_q^e\tilde{\Phi}_m(\varphi). \tag{3.1}$$

Here H_q^e is a Hamiltonian with a certain gauge potential and is transformed into $\tilde{H}_q^e$.

Let us determine the form of H_q^e, that is, how to introduce the gauge potential A_ϑ into H_q. Unfortunately, it is difficult to solve for all $\Gamma_\alpha\in U(1)$. Therefore, we restrict the gauge group $U(1)$ to $U'(1)$, where $U'(1)$ is generated by

$$\Gamma'_\lambda=\exp\left(\mathrm{i}\frac{e}{c\hbar}\lambda\varphi\right)$$

with λ a real constant. The first possibility for introducing the gauge interaction may be given by the following replacement in the Hamiltonian H_q:

$$\begin{aligned} p_\varphi^{(-)}&\to p_\varphi^{(-)}+\mathrm{i}\frac{e}{c}A_\varphi^{(-)}\,\mathrm{e}^{-\mathrm{i}\varphi}q^{-\mathrm{i}\partial_\varphi}\\ p_\varphi^{(+)}&\to p_\varphi^{(+)}-\mathrm{i}\frac{e}{c}A_\varphi^{(+)}\,q^{\mathrm{i}\partial_\varphi}\mathrm{e}^{\mathrm{i}\varphi}. \end{aligned} \tag{3.2}$$

Under these replacements, the requirement (3.1) holds, if the gauge potentials transform as

$$\begin{aligned} A_\varphi^{(-)}&\xrightarrow{\Gamma'_\lambda}\tilde{A}_\varphi^{(-)}=q^{-e\lambda/c\hbar}\left\{A_\varphi^{(-)}+\left(\frac{c\hbar(q)}{e}\right)\frac{1-q^{e\lambda/c\hbar}}{1-q}\right\}\\ A_\varphi^{(+)}&\xrightarrow{\Gamma'_\lambda}\tilde{A}_\varphi^{(+)}=q^{e\lambda/c\hbar}\left\{A_\varphi^{(+)}+\left(\frac{c\hbar^*(q)}{e}\right)\frac{1-q^{-e\lambda/c\hbar}}{1-q^{-1}}\right\}. \end{aligned} \tag{3.3}$$

In the limit $q\to 1$, both of $\tilde{A}_\varphi^{(\pm)}$ reduce to $A_\varphi+(\mathrm{d}/\mathrm{d}\varphi)\alpha_q(\varphi)$, known as the gauge transformation of the second kind, for $\alpha_q(\varphi)=\lambda\varphi$. For simplicity, we choose the gauge fields as $A_\varphi^{(\pm)} = \mp i A_0^{(\pm)}$ with real constants $A_0^{(\pm)}$. Of course, $A_0^{(\pm)}$ may have q-dependence and we claim that $\lim_{q\to 1} A_0^{(\pm)} = A_0$. Using the replacements given in (3.2), we have Hamiltonian with the gauge interaction.

$$H_q^e = \kappa\left\{p_\varphi^{(+)}p_\varphi^{(-)} - \frac{e}{c}\left(h^*(q)A_0^{(-)}\frac{q^{-i\partial_\varphi}-1}{1-q^{-1}} + h(q)A_0^{(+)}\frac{q^{i\partial}-1}{1-q}\right) + \left(\frac{e}{c}\right)^2 A_0^{(-)}A_0^{(+)}\right\}. \tag{3.4}$$

In the limit $q\to 1$, H_q^e is reduced to that for a rotator in a constant magnetic field A_0 perpendicular to the rotator's plane, that is

$$H_q^e \overset{q\to 1}{\to} \kappa\left(p_\varphi - \frac{e}{c}A_0\right)^2$$

where $p_\varphi = -i\hbar\partial/\partial\varphi$. It is easily seen that eigenfunctions for H_q^e are the same functions $\Phi_m(\varphi)$ given in (2.11) and eigenvalues for $\Phi_m(\varphi)$ are obtained as follows:

$$\begin{aligned}
E_m^e &= E_m - \frac{e}{c}\kappa\left(h^*(q)A_0^{(-)}e^{\alpha/2(m-1)} + h(q)A_0^{(+)}e^{-\alpha/2(m+1)}\right)\frac{\sinh(\alpha m/2)}{\sinh(\alpha/2)} \\
&\quad + \kappa\left(\frac{e}{c}\right)^2 A_0^{(+)}A_0^{(-)} \qquad \text{for } q = e^{\alpha} \\
E_m^e &= E_m - \frac{e}{c}\kappa\left(h^*(q)A_0^{(-)}e^{i\alpha/2(m+1)} + h(q)A_0^{(+)}e^{-i\alpha/2(m+1)}\right)\frac{\sin(\alpha m/2)}{\sin(\alpha/2)} \\
&\quad + \kappa\left(\frac{e}{c}\right)^2 A_0^{(+)}A_0^{(-)} \qquad \text{for } q = e^{i\alpha}
\end{aligned} \tag{3.5}$$

with $\alpha \in \boldsymbol{R}\backslash\{0\}$. The magnetic field dissolves the degeneracy (2.14), i.e. $E_m^e \neq E_{-m}^e$. This is just a q-deformed version of Zeeman's effect. Indeed, with $\kappa = 1/2I$

$$E_m^e \overset{q\to 1}{\to} \frac{\hbar^2}{2I}\left(m - \frac{eA_0}{ch}\right)^2 \tag{3.6}$$

which are the energy eigenvalues for a rotator around a magnetic flux $(1/2\pi)A_0$. In contrast, the $\boldsymbol{Z}_k$ symmetry in the case $q = e^{2\pi i(1/k)}$ still remains.

We next consider another possibility for introduction of gauge interaction, which is performed by the following replacements:

$$\begin{aligned}
p_\varphi^{(-)} &\to p_\varphi^{e(-)} = -ih(q)e^{-i\varphi}\frac{1-q^{-i\partial_\varphi-(e/ch)A_\varphi^{(-)}}}{1-q} \\
p_\varphi^{(+)} &\to p_\varphi^{e(+)} = ih^*(q)\frac{1-q^{i\partial_\varphi+(e/ch)A(+)_\varphi}}{1-q^{-1}}e^{i\varphi}.
\end{aligned} \tag{3.7}$$

In general, $p_\varphi^{e(-)}$ does not satisfy the commutation relation (2.1) with $e^{i\varphi}$. The relation is, however, preserved for the choice $A_\varphi^{(\pm)} = A_0^{(\pm)} = \text{constant}$. Moreover, in this choice, the transformation rules for $A_0^{(\pm)}$ under $U'(1)$ are easily solved and obtained as

$$A_0^{(\pm)} \overset{\Gamma_\lambda'}{\to} \tilde{A}_0^{(\pm)} = A_0^{(\pm)} + \lambda. \tag{3.8}$$

And the Hamiltonian with the gauge interaction is given by

$$H_q^e = \kappa|h(q)|^2\frac{2 - q^{i\partial_\varphi+(e/ch)A_0^{(+)}} - q^{-i\partial_\varphi-(e/ch)A_0^{(-)}}}{(1-q)(1-q^{-1})}. \tag{3.9}$$

The Hamiltonian also has the same eigenfunctions $\Phi_m(\varphi)$, and eigenvalues thereof are derived as follows:

$$E_m^e = \kappa |h(q)|^2 \frac{2 - q^{-m+(e/c\hbar)A_0^{(+)}} - q^{m-(e/c\hbar)A_0^{(-)}}}{(1-q)(1-q^{-1})}. \tag{3.10}$$

As in the first case, the degeneracy between E_m and E_{-m} has been dissolved, but that for the Z_k symmetry in the case with $q = e^{2\pi i(1/k)}$ has not. The energy eigenvalue has the same $q \to 1$ limit as (3.6) if $A_0^{(\pm)} \xrightarrow{q\to 1} A_0$. When we choose $A_0^{(+)} = A_0^{(-)} = A$, we can observe a remarkable fact. For the Hamiltonian with the gauge interaction (3.9), the same energy eigenvalues (2.10) for the original Hamiltonian H_q are obtained if we make use of eigenfunctions

$$\Phi_m^e(\varphi) = e^{i(e/c\hbar)A\varphi}\Phi_m(\phi). \tag{3.11}$$

These eigenfunctions satisfy the boundary conditions

$$\Phi_m^e(\varphi + 2\pi) = e^{2\pi i(e/c\hbar)A}\Phi_m^e(\varphi). \tag{3.12}$$

If we select the gauge potential A such that $(e/c\hbar)A = (1/n)$ $(n = 2, 3, \ldots)$, we see that the system with the Hamiltonian (3.9) represents that for a particle obeying the Fermi statistics ($n = 2$) or other fractional statistics ($n \geqslant 3$), of which the energy level is given by (2.10).

4. Discussions of algebraic structure

In this section we will investigate algebraic structures of this model. As we have mentioned in section 2, this model has a Z_k symmetry when the deformation parameter q is taken to be $q = \exp(2\pi i(1/k))$. We will show in the first subsection that the symmetry is related to the symmetry under the infinite-dimensional algebra, $W_{1+\infty}$ [11] without central extension. Furthermore, $U_q(sl_2)$ is found in subsection 4.2.

4.1. $w_{1+\infty}$ and $W_{1+\infty}$

We first investigate a classical theory of the model discussed in section 2. For the time being, we do not restrict q to be a root of unity. The classical theory is obtained by dropping $h(q)$ and replacing $-i\partial_\varphi \to \xi \in R$, where ξ corresponding to momentum, and the angle variable φ obey the following Poisson bracket

$$\{\varphi, \xi\}_{PB} = 1. \tag{4.1}$$

Then the phase space M is a cylinder, $M = S^1 \times R$, and physical observables are real-valued functions on M. The Hamiltonian in the classical theory can be written as

$$H_q^c = \kappa \frac{2 - q^{\xi} - q^{-\xi}}{(1-q)(1-q^{-1})}. \tag{4.2}$$

The Hamiltonian H_q^c is invariant under canonical transformations generated by $j_n^r \overset{\text{def}}{=} e^{in\varphi}\xi^{r+1}$

$$\{H_q^c, j_n^r\}_{PB} = 0. \tag{4.3}$$

The generators $j_n^r (n \in Z, r = -1, 0, 1, \ldots)$ form the algebra $w_{1+\infty}$ as follows:

$$\{j_n^r, j_m^s\}_{PB} = i(n(s+1) - m(r+1))j_{n+m}^{r+s}. \tag{4.4}$$

This is known as the algebra of area preserving diffeomorphism of a cylinder.

Let us return to the quantum mechanics with the Hamiltonian H_q given in (2.7). According to the quantization, the momentum ξ in j^r_n must be replaced by $-\mathrm{i}\partial_\varphi$ and we define new generators as $\hat{j}^r_n \stackrel{\text{def}}{\to} \mathrm{e}^{\mathrm{i}n\varphi}(-\mathrm{i}\partial_\varphi)^{r+1}$, $n \in Z$, $r=-1, 0, 1, \ldots$. Note that the generators $\hat{j}^r_n$ no longer obey the commutation relation (4.4) after the replacement $\mathrm{i}\{,\}_{PB} \to [,]$. On the other hand, taking account of the relation $q^{-\mathrm{i}\partial_\varphi}\mathrm{e}^{\mathrm{i}\varphi} = q\mathrm{e}^{\mathrm{i}\varphi}q^{-\mathrm{i}\partial_\varphi}$, the following commutation relation is obtained:

$$\kappa|h(q)|^2 \frac{2-q^{\mathrm{i}\partial_\varphi}-q^{-\mathrm{i}\partial_\varphi}}{(1-q)(1-q^{-1})}\hat{j}^r_n = \hat{j}^r_n \kappa|h(q)|^2 \frac{2-q^{-n}q^{\mathrm{i}\partial_\varphi}-q^{n}q^{-\mathrm{i}\partial_\varphi}}{(1-q)(1-q^{-1})}. \tag{4.5}$$

Thus, for arbitrary q, the Hamiltonian in the quantum mechanical case is not necessarily commutable with the generator $\hat{j}^r_n$. A special case occurs when q is a root of unity, i.e. $q=\exp(2\pi\mathrm{i}(1/k))$, $k=2, 3, \ldots$. In this case the Hamiltonian H_q satisfies the following commutation relation:

$$[H_q, J^r_n]=0 \tag{4.6}$$

where we have used $J^r_n \equiv k^{-r}\hat{j}^r_{nk}$. The commutation relations among J^r_n are the $W_{1+\infty}$ without central extension

$$[J^r_n, J^s_m] = \sum_{\mu=0}^{\max(r,s)} \left[\binom{r+1}{\mu+1} m^{\mu+1} - \binom{s+1}{\mu+1} n^{\mu+1} \right] J^{r+s-\mu}_{n+m}. \tag{4.7}$$

It should be noticed here that the classical theory has the symmetry $w_{1+\infty}$ for arbitrary deformation parameter $q=\mathrm{e}^{\alpha}$, or $\mathrm{e}^{\mathrm{i}\alpha}$ for all $\alpha \in \boldsymbol{R}$, while, in the quantum mechanical system, the parameter q must be quantized if we require the commutation relation (4.6). The Z_k symmetry is understood that the eigenfunctions $\Phi_m(\varphi)$ and $\Phi_{m+kn}(\varphi)$ are related by the generator J^r_n such that

$$J^r_n \Phi_m(\varphi) = m^{r+1}\Phi_{m+kn}(\varphi). \tag{4.8}$$

The Hilbert space of the model with $q=\exp(2\pi\mathrm{i}/k)$ reduces to a finite-dimensional space after dividing the full space generated by all functions Φ_m by the symmetry.

4.2. $U_q(sl_2)$ in this model

In section 2 we have given the commutation relation between $\mathrm{e}^{\mathrm{i}\varphi}$ and $p^{(-)}_\varphi$ only. In addition to $p^{(-)}_\varphi$ we have introduced another operator $p^{(+)}_\varphi$, but we did not discuss the commutation relation between $p^{(-)}_\varphi$ and $p^{(+)}_\varphi$ there. To calculate the relation and also to derive an algebra among them are the aims of this subsection.

In the following discussions it is convenient to define operators $X_\pm$ as

$$\begin{aligned} X_+ &\stackrel{\text{def}}{=} \frac{1}{\mathrm{i}h^*(q)}p^{(+)}_\varphi = \frac{1-q^{\mathrm{i}\partial_\varphi}}{1-q^{-1}}\mathrm{e}^{\mathrm{i}\varphi} \\ X_- &\stackrel{\text{def}}{=} \frac{1}{\mathrm{i}h(q)}p^{(-)}_\varphi = -\mathrm{e}^{-\mathrm{i}\varphi}\frac{1-q^{-\mathrm{i}\partial_\varphi}}{1-q}. \end{aligned} \tag{4.9}$$

The calculation of the commutation relation between X_+ and X_- is straightforward, and we obtain

$$[X_+, X_-] = \frac{K^2 - K^{-2}}{q^{1/2} - q^{-1/2}} \tag{4.10}$$

where

$$K \overset{\text{def}}{=} q^{1/2(-i\partial_\varphi + 1/2)}. \tag{4.11}$$

The commutation relations among K and $X_\pm$ are easily obtained as follows:

$$KX_\pm = q^{\pm 1/2} X_\pm K. \tag{4.12}$$

These relations (4.10) and (4.12) are nothing but the commutation relations of $U_q(sl_2)$.

We next investigate its highest weight representation. The actions of $X_\pm$ on the eigenfunctions are

$$\begin{aligned} X_+\Phi_m(\varphi) &= q^{-1/2m}[m+1]_q \Phi_{m+1}(\varphi) \\ X_-\Phi_m(\varphi) &= -q^{1/2(m-1)}[m]_q \Phi_{m-1}(\varphi) \end{aligned} \tag{4.13}$$

where q-integers are taken as $[x]_q = (q^{x/2} - q^{-x/2})/(q^{1/2} - q^{-1/2})$. Noticing that $X_-\Phi_0 = 0$, we can expect that the highest weight module can be constructed on the state Φ_0 by acting on X_+. We define weight vectors $\Psi_m(\varphi)$ as

$$\Psi_m(\varphi) = \frac{(X_+)^m}{[m]_q!} \Phi_0(\varphi) \qquad m = 0, 1, \cdots. \tag{4.14}$$

Let V be the highest weight U_q-module defined as $V = \{\Psi_m(\varphi) | m \in Z_{\geqslant 0}\}$. Taking into account that $X_-\Psi_0(\varphi) = 0$ and $K\Psi_0(\varphi) = q^{1/4}\Psi_0(\varphi)$, V is the highest weight module with the highest weight state Ψ_0 of dimension $\frac{1}{2}$. The actions of $X_\pm$ and K on V are given by

$$\begin{aligned} X_+\Psi_m(\varphi) &= [m+1]_q \Psi_{m+1}(\varphi) \\ X_-\Psi_m(\varphi) &= -[m]_q \Psi_{m-1}(\varphi) \\ K\Psi_m(\varphi) &= q^{1/2(m+1/2)} \Psi_m(\varphi). \end{aligned} \tag{4.15}$$

The highest weight module is irreducible unless q is a root of unity.

We finally notice that our model with the Hamiltonian (2.7) is not invariant under the actions of $U_q(sl_2)$, because the Hamiltonian can be written as $H_q = \kappa |h(q)|^2 X_+X_-$ which is not the Casimir operator of $U_q(sl_2)$. The quadratic Casimir invariant of $U_q(sl_2)$

$$\mathscr{C} = X_+X_- + \frac{q^{-1/2}K^2 + q^{1/2}K^{-2} - 2}{(q^{1/2} - q^{-1/2})^2}$$

is just zero when we express the operators X_+, X_- and K in terms of (4.9) and (4.11).

5. Extension for many-angle variables

Let us study the introduction of many-angle variables, $\varphi_1, \varphi_2, \cdots, \varphi_N$ ($\forall\varphi_i: 0 \leqslant \varphi_i < 2\pi$). Here we investigate an extension realizing the symmetry among all these variables. We can derive the following expression for the momentum:

$$\mathscr{P}_r^{(-)} = q^{-i\Sigma_{j\neq r}\partial_j} p_r^{(-)} \qquad (r = 1, 2, \cdots, N) \tag{5.1}$$

where $\partial_j = \partial_{\varphi_j}$ and $p_r^{(-)}$ are defined by the replacement of φ and ∂_φ, respectively, with φ_r and ∂_{φ_r} in $p_\varphi^{(-)}$. The commutation relations are given as follows:

$$\begin{aligned}
&[e^{i\varphi_r}, \mathcal{P}_s^{(-)}]_q = i\delta_{rs} h(q) q^{-i\Sigma_{j\neq r}\partial_j} \\
&[e^{i\varphi_r}, e^{i\varphi_s}] = 0 \\
&[\mathcal{P}_r^{(-)}, \mathcal{P}_s^{(-)}] = 0.
\end{aligned} \tag{5.2}$$

Note that the commutation relations are symmetric with respect to the indices. The Hamiltonian can be written as

$$\mathcal{H}_q = \kappa \sum_{r=1}^{N} \mathcal{P}_r^{(+)} \mathcal{P}_r^{(-)} \tag{5.3}$$

where $\mathcal{P}_r^{(+)} = q^{i\Sigma_{j\neq r}\partial_j} p_r^{(+)}$. In the limit $q \to 1$, all the commutation relations (5.2) become classical numbers and the Hamiltonian reduces to

$$\mathcal{H}_q \xrightarrow{q\to 1} -\kappa\hbar^2 \sum_{r=1}^{N} \frac{\partial^2}{\partial\varphi_r^2}. \tag{5.4}$$

We see that, by replacing the variables φ_r by the coordinates x_r with periodic boundary conditions, this extension naturally reproduces the Hamiltonian for a free particle in N-dimensional space in the limit $q \to 1$.

6. Some examples with potentials

Let us study some examples of the present theory.

6.1. *Two-dimensional problems*

Let us study the q-deformed version of two-dimensional problems represented by the Hamiltonian which is written in terms of the cylindrical coordinates r and φ as

$$H = -\frac{\hbar^2}{2M}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\right] + V(r). \tag{6.1}$$

Since the eigenfunctions for $\partial^2/\partial\varphi^2$ are the same as those of $p_\varphi^{(+)} p_\varphi^{(-)}$, we may replace $\partial^2/\partial\varphi^2$ by $-p_\varphi^{(+)} p_\varphi^{(-)}/\hbar^2$ in H. Note that in the limit $q \to 1$ both Hamiltonians coincide with each other. The equation for the variable r is derived as

$$\left[-\frac{\hbar^2}{2M}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{\Lambda_m}{r^2}\right) + V(r)\right] R_n^m(r) = E_{nm} R_n^m(r), \tag{6.2}$$

where $\psi_{nm}(r,\varphi) = R_n^m(r)\Phi_m(\varphi)$ and

$$\Lambda_m = |h(q)|^2 [m]_q^2 / \hbar^2. \tag{6.3}$$

For the two-dimensional harmonic oscillator described by $V(r) = M\omega^2 r^2/2$ we immediately derive the energy eigenvalues

$$E_{nm} = \hbar\omega(n + \gamma_m + 1), \tag{6.4}$$

where $n \in Z_{\geq 0}$ and $\gamma_m = \sqrt{\Lambda_m}$. Since γ_m is not generally an integer, excitation energies for the φ motions are not represented by $\hbar\omega$ times integers. For the choice of $q = \exp(i2\pi/k)$ the infinite degeneracies again appear.

6.2. *q-deformed Coulomb interaction*

We can introduce the q-deformed motion to the azimuthal angle part of the three-dimensional problems. Let us study a model with Coulomb potential ($V(r) = -Ze^2/r$ and $Z > 0$). By using the following coordinates (ξ, η and φ) defined as

$$x = \xi\eta \cos\varphi \qquad y = \xi\eta \sin\varphi \qquad \text{and } z = (\xi^2 - \eta^2)/2$$

the q-deformed Hamiltonian may be written down as

$$H_q = -\frac{\hbar^2}{2M}\frac{1}{\xi^2+\eta^2}\left[\frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial}{\partial\xi}\right) + \frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial}{\partial\eta}\right) - \left(\frac{1}{\xi^2}+\frac{1}{\eta^2}\right)\frac{p_\varphi^{(+)}p_\varphi^{(-)}}{\partial^2}\right] - \frac{2Ze^2}{\xi^2+\eta^2}. \tag{6.5}$$

Taking the eigenfunction as

$$\psi_m(\xi, \eta, \varphi) = f(\xi)g(\eta)\Phi_m(\varphi)$$

we can solve the equation $H_q\psi = E\psi$ straightforwardly and obtain the energy eigenvalues as

$$E_{N_m} = -\frac{MZ^2e^4}{2\hbar^2N_m^2}, \tag{6.6}$$

where $N_m = N + \gamma_m + 1$ for $N \in Z_{\geq 0}$. Except the special case where γ_m is an integer, the energy spectrum is different from the well known spectrum for Coulomb potentials. In particular the infinite degeneracy for every energy level again appears for $q = \exp(i2\pi/k)$.

7. Remarks

We have studied some extensions and applications of quantum mechanics with the q-deformed commutator presented in [10]. We have shown that when the deformation parameter is a root of unity, infinite degeneracy appears in each energy level, and then the number of energy levels is finite. The origin of the degeneracy is that the model has the symmetry of infinite dimension, $W_{1+\infty}$, in the case $q = \exp(i2\pi/k)$, $k = 2, 3, \ldots$. We have, further, pointed out that the classical theory of this model has the symmetry, $w_{1+\infty}$, for arbitrary q. The remarkable point is that the symmetry, $w_{1+\infty}$, of the classical system is broken by means of the quantization but is recovered, only when $q = \exp(i2\pi/k)$ ($k = 2, 3, \ldots$) is chosen, as the symmetry $W_{1+\infty}$. In other words, the parameter q is quantized according to the quantization of the model. We have seen that the infinite degeneracy for every energy level appears in all applications presented in this paper. It should be remarked that the degeneracy is dissolved for the general choice of the gauge function A_φ [10]. As was noted for the discretized rings [10], not only bosons but fermions can condense in every energy level as well because of the infinite degeneracy. At present the model we presented here is of theoretical interest only. Though we need more careful considerations and detailed investigations before we conclude whether such phenomena are observable in realistic processes, it will be interesting to look for such phenomena.

We remark that q-deformed interactions can be introduced perturbatively. Since the eigenfunctions are the same as those for the azimuthal angle, we can solve them easily. Taking into account that $p_\varphi^{(-)}$ generates a finite displacement such as $\varphi \to \varphi + \alpha$ for the choice $q = e^{i\alpha}$, we may also consider that the Hamiltonian (2.7) represents an equation for difference calculus. Extensions to difference equations such as equations for lattices are interesting, but we do not discuss them here.

References

[1] Biedenharn L C 1989 *J. Phys. A: Math. Gen.* **22** L873
[2] Macfarlane A J 1989 *J. Phys. A: Math. Gen.* **22** 4581
[3] Odaka K, Kishi T and Kamefuchi S 1991 *J. Phys. A: Math. Gen.* **24** L591
[4] Wess J and Zumino B 1990 *Nucl. Phys.* B (*Proc. Suppl.*) **18** 302
[5] Zumino B 1991 *Mod. Phys. Lett.* A **16** 1225
[6] Aref'eva I Ya and Volovich I V 1991 *Phys. Lett.* **268 B** 179
[7] Carow-Watamura U, Schlieker M and Watamura S 1991 *Z. Phys.* C **49** 439
[8] Schwenk J and Wess J 1992 *Phys. Lett.* **291 B** 273
[9] Li You-quan and Sheng Zheng-mao 1992 *J. Phys. A: Math. Gen.* **25** 6779
[10] Kobayashi T and Suzuki T 1993, *Phys. Lett.* B in press
[11] Pope C N, Romans L J and Shen X 1990 *Phys. Lett.* **242 B** 401
[12] Suzuki T 1993 *J. Math. Phys.* in press